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AN

EXAMINATION OF SOME QUESTIONS
CONNECTED WITH
GAMES OF CHANCE.

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EXAMINATION OF SOME QUESTIONS

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THE questions which I propose to examine in the following paper, although not themselves dependent on chance, have arisen entirely from games in which it predominates. To determine some method of betting upon a number of successive events, (and the probability of each of which is either equal to, or less than one-half,) by which a profit shall be realised after a considerable number of them have been decided, is a problem which has occupied the attention, and exhausted the efforts, of one set of speculators, as completely as that of the quadrature of the circle has defeated the labours of another. The first and most simple plan, is that of doubling the stake whenever a loss occurs. This is well known, and has been so frequently practised, as to have acquired a peculiar name; it is technically called the *martingal;* it requires for its success, that the person who employs it have the power of leaving off whenever he please, and that he have the command of an unlimited capital. If the chance of the events happening is one-third instead of one-half, the stake must be tripled. Supposing a player adopt this system in betting on a number of independent events, the chance of occurring of each of which is one-half, he would naturally desire to know, if possible, before he began, what would be his profit or loss, supposing p of the events decided in his favour, and q against him. In this particular plan of playing, it so happens, that he cannot, from the mere knowledge of the number of favourable and of unfavourable events, arrive at the conclusion he desires. I shall presently show, that these data alone are insufficient, and that, in order to determine the question, not merely the number, but the order of succession must be given.

It is probable that this difficulty, in the case of most frequent occurrence, has deterred many from attempting other similar problems. Indeed, on the first view of such questions, it is by no means apparent, that any of them can be solved without reference to the order in which the events take place. The mode of inquiry which I shall point out, will show that in many cases their mutual arrangement is not required amongst the data, and will furnish a criterion by which we may determine, in any given case, whether it is necessary.

I shall first examine the case of the martingal, which, although the results it leads to are of a negative nature, will introduce us to the method of treating these questions, and from its frequent practice is rather interesting.

Let us suppose a gamester bet a certain sum $2u$, upon an event whose chance of happening is one-half; whenever he wins he repeats the same bet; but whenever he loses, he doubles his last stake. If he should win p and lose q times, it is required to ascertain how much he will have won or lost.

The first stake being $2u$, which may either be gained or lost, we may represent the gamester's profit after one event is decided by $2u(-1)^a$, a being any whole number; for since the nature of the number a is left undecided, whether it is an even or an odd one, the expression just given will represent either a profit or a loss. First, if a is an even number, his next stake remains the same as it was before, or equal to $2u$, if it is an odd number, his second stake must be $4u$. We must therefore endeavour to find some function of a which shall be equal to $2u$, when a is an even number, and become $4u$, when a is an odd one. A great variety of functions may be found satisfying this condition; such are $2u \times 2^{\frac{1-(-1)^a}{2}}$, $2u\left(\frac{3+\cos a\pi}{2}\right)$, &c. and many others. Our choice amongst the infinite variety which present themselves, must be directed by the ulterior operations of the gamester; and, first, I remark, that every stake must be equal to the constant quantity u, multiplied by some power of 2. The next remark which may guide us in this choice of a proper function is, that the index of that power of 2 is determined by the number of times (in immediate succession) the event has been unfavourable, reckoning back from the event about which the stake is proposed: if, therefore, at any period, all the preceding determinations have been unfavourable, the amount of the next stake will be influenced by them all. It may also be noticed, that the successive powers of 2 can be formed by the addition of all the preceding ones, together with unity, thus,

$$1,\ 2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5$$

$1+2^0=2$, $1+2^0+2^1=4$, $1+2^0+2^1+2^2=8$, and so on:

From these considerations it appears, that the function we require may consist of a series of functions, each multiplying the successive powers of 2. In order to determine these, let us suppose that at some point of the game, the last event has proved favourable; then, by the conditions, the next stake is $2u$; and whatever be the course of succeeding events, $2u$ will always form part of the stake; therefore it need not be multiplied by any function of a, b, c, &c. the letters which determine the winning or losing of the subsequent events. We may therefore assume $2u$ as the constant part of every stake, without reference to any particular order in their occurrence. If the failure or happening of this event is represented by $(-1)^a$, a being an odd number in the first, and an even number in the second case, we must multiply the next power of 2, or 2^1, by some function of a which shall vanish when a is an even number, and become unity when it is an odd one. Such a function is easily found, and one of the simplest is $\frac{1-(-1)^a}{2}$. The next stake is therefore $u\,(2+\frac{1-(-)^a}{2}\,2)$, whatever be the form of a.

The failing or happening of this event may be represented by $(-1)^b$, and the profit of the player by this event is then represented by $u\,(2+\frac{1-(-1)^a}{2}\,2^1)\,(-1)^b$.

The third stake must comprehend the second power of 2, and will be of the form

$$u\left(2+2^1 f(a,b)+2^2 f_1(a,b)\right).$$

If b is an even number, or the second event is favourable, in that case, the new stake would be $2u$; and therefore both $f(a,b)$ and $f_1(a,b)$ must vanish. This will take place if each has a factor of the form $\frac{1-(-1)^b}{2}$, and the new stake in consequence becomes

$$u\left\{2+2^1\frac{1-(-1)^b}{2}f(a,b)+2^2\frac{1-(-1)^b}{2}f_1(a,b)\right\};$$

f and f_1 not being the same functions as before. If a had been an even number, we may consider the first bet as not having been made, since it has no influence on the succeeding ones; and in this case the expression ought to reduce itself to

$$u\left\{2+2^1\frac{1-(-1)^b}{2}\right\};$$

$f(a,b)$ must therefore equal unity, and $f_1(a,b)$ must vanish when a is an even number. This gives

$$u\left\{2+2^1\frac{1-(-1)^b}{2}+2^2\frac{1-(-1)^b}{2}f_1(a,b)\right\}.$$

This expression is reduced to $2u$ if b is an even number, and to

$$4u+4uf_1(a,b)$$

when b is an odd one; $f_1(a,b)$ must therefore be such a function of a, that when a is odd, it shall become unity, and when even, equal zero; $\frac{1-(-1)^a}{2}$ is such a function, and we then have for the third stake

$$u\left\{2+2\frac{1-(-1)^b}{2}+2^2\frac{1-(-1)^b}{2}\cdot\frac{1-(-1)^a}{2}\right\}.$$

The law by which we may represent the stake to be ventured, after the determination of any number of events, is now apparent; had it not been sufficiently so, the same reasoning which has been already explained at some length would have assigned it for the fourth stake,

$$u\left\{2+2\frac{1-(-1)^c}{2}+2^2\frac{1-(-1)^b}{2}\cdot\frac{1-(-1)^c}{2}+2^3\frac{1-(-1)^a}{2}\cdot\frac{1-(-1)^b}{2}\cdot\frac{1-(-1)^c}{2}\right\}.$$

We can now represent the profit or loss on each event, without determining which of the two it is:

$$\text{On the 1st he gains } u\left\{2\right\}(-1)^{a}$$

$$\text{2d} \qquad u\left\{2+\overline{1-(-1)}^{a}\right\}(-1)^{b}$$

$$\text{3d} \qquad u\left\{2+\overline{1-(-1)}^{b}+\overline{1-(-1)}^{a}\,.\,\overline{1-(-1)}^{b}\right\}(-1)^{c}$$

$$\text{4th} \qquad u\left\{\begin{array}{l}2+\overline{1-(-1)}^{c}+\overline{1-(-1)}^{b}\,.\,\overline{1-(-1)}^{c}+\overline{1-(-1)}^{a} \\ \qquad .\,\overline{1-(-1)}^{b}\,.\,\overline{1-(-1)}^{c}\end{array}\right\}(-1)^{d}$$

$$\text{5th} \qquad u\left\{\begin{array}{l}2+\overline{1-(-1)}^{d}+\overline{1-(-1)}^{c}\,.\,\overline{1-(-1)}^{d}+\overline{1-(-1)}^{b}\,.\,\overline{1-(-1)}^{c}\,.\,\overline{1-(-1)}^{b} \\ \qquad +\overline{1-(-1)}^{a}\,.\,\overline{1-(-1)}^{b}\,.\,\overline{1-(-1)}^{c}\,.\,\overline{1-(-1)}^{d}\end{array}\right\}.\,(-1)^{e}$$

&c. &c. &c.

The sum of all these is the object sought: with respect to that part which multiplies $2u$, it is easily found; it is

$$2u\left\{(-1)^{a}+(-1)^{b}+(-1)^{c}+\ldots\text{\&c.}\right\};$$

but we know that he has gained p, and lost q times; consequently p of the quantities a, b, c, &c. are even numbers, and q of them odd ones; therefore the sum of the series, multiplying $2u$, is $p-q$, and the value of the part alluded to is

$$2(p-q)u.$$

So far, then, we can proceed by means of the data furnished in the question; but they are insufficient for its complete solution, on attempting to find the sum of that part of the expression in which (-1) has only one index, it appears to be

$$u\left\{1(-1)^{b}+2(-1)^{c}+3(-1)^{d}+\&c.\right\};$$

and the knowledge that p of the letters a, b, c, &c. represent even, and q of them odd numbers, will not assist us in determining the sum of this series; we must also be acquainted with the order of succession of the even and odd numbers. Another reason will afterwards be assigned, why it is not sufficient merely to be acquainted with the number of favourable and of unfavourable events, which may with more propriety be stated, when some other questions of a similar nature have been examined.

Let us now consider another case, in which the amount of the sum staked on each event follows a different law, we will suppose the following problem:

A gamester begins a series of bets on an event whose chance of occurring is one-half, by staking the sum u. Whenever he wins he makes the next succeeding stake less than his last by the quantity v; but if he lose, he then increases his stake by the same quantity. Supposing he should win p times, and lose q times, what will he have gained or lost on the whole number $p+q$.

His first stake being u, his first profit may be expressed by $u(-1)^{a}$, which will be won or lost according as u is even or odd; in the first case his next stake would be $u-v$, and in the second it would be $u+v$. These two cases may be combined into one, and thus expressed $u-v(-1)^{a}$, and his second profit will be $\left\{u-v(-1)^{a}\right\}(-1)^{b}$; his next stake will be $u-v(-1)^{a}-v$ if b is even; but it will be $u-v(-1)^{a}+v$

if that number is odd. These may be represented thus: $u - v\left\{(-1)^a + (-1)^b\right\}$, and his third profit will be

$$\left[u - v\left\{(-1)^a + (-1)^b\right\}\right](-1)^c;$$

the succeeding stakes are easily represented in the same manner, and his profits stand thus:

$$u(-1)^a$$
$$u(-1)^b - v\left\{(-1)^a\right\}(-1)^b$$
$$u(-1)^c - v\left\{(-1)^a + (-1)^b\right\}(-1)^c$$
$$u(-1)^d - v\left\{(-1)^a + (-1)^b + (-1)^c\right\}(-1)^d$$
$$u(-1)^e - v\left\{(-1)^a + (-1)^b + (-1)^c + (-1)^d\right\}(-1)^e,$$

&c. &c.

the value of the part multiplying u, or

$$(-1)^a + (-1)^b + (-1)^c + \&c.$$

is easily found, since we know that p of the quantities $a, b, c,$ are even numbers, and q of them odd numbers. The first part of the expression is therefore

$$(p - q)\,u.$$

In order to determine the second part, we must observe, that the expression

$$(a')\,b' + (a' + b')\,c' + (a' + b' + c')\,d' + (a' + b' + c' + d')\,e' +, \&c.$$

is equal to the sum of all the combinations, two by two of the quantities a', b', c', &c.; and, moreover, that if these latter are the roots of an equation, then the expression just written is equal to the co-efficient of its third term. If we change a' into $(-1)^a$, b' into $(-1)^b$, c' into $(-1)^c$, &c. it becomes the same as the one whose value we are seeking. Hence then it appears, that the sum of all the quantities, multiplying v, is equal to the co-efficient of the third term of an equation whose roots are $(-1)^a$, $(-1)^b$, $(-1)^c$, &c. or of the equation

$$o = x - (-1)^a \,.\, x - (-1)^b \,.\, x - (-1)^c \,.\,.\,.\,.$$

but we know that p of the quantities a, b, c, &c. are even numbers, and q of them odd ones; therefore this equation has p equal roots of the form $+1$, and q equal ones of the form -1, or the equation is

$$o = (x-1)^p \; (x+1)^q\,;$$

and the quantity which multiplies v, is equal to the co-efficient of the third term of this expression, which is

$$\frac{p \,.\, p-1}{1\,.\,2} - \frac{p}{1} \,.\, \frac{q}{1} + \frac{q \,.\, q-1}{1\,.\,2} = \frac{(p-q)^2 - (p+q)}{2}\,;$$

so that the profit of the gamester is

$$W = (p-q)\,u - v\,\frac{(p-q)^2 - (p+q)}{2}. \qquad (1)$$

This result is entirely independent of the order in which the events occurred; and we may learn from the method that has been employed for its solution, that whenever the sum of all the winnings or losings is a symmetrical function of the quantities $(-1)^a$, $(-1)^b$, $(-1)^c$, &c. the final conclusion will not depend on the order in which the events succeed each other, but on the actual number of favourable and unfavourable events.

In the instance we are now considering, if the number of successful and of unsuccessful cases are equal, the gamester who adopts this system of play will always win, for in that case $p = q$, and $W_1 = p v$; as a numerical example of the formula just investigated. Let us suppose

A person stakes 100 shillings on the event of a piece of money thrown into the air falling with one of its faces uppermost, in preference to the other; whenever he wins he diminishes his stake five shillings, and whenever he loses he increases it by the same sum. Supposing he makes 800 successive bets, and wins p, and loses q of them.

If p and q are equal, each being 400, his profit will be $400 \times 5 = 2000$ shillings.

From this it appears, that he will win even though a smaller number than one-half of the events prove favourable. In order to determine how many times he must win, that he may neither lose nor gain on the whole number, we must make the value of W equal to zero, putting $p + q = a$; this gives for the required value of p

$$p = +\tfrac{1}{2}\left(a + \tfrac{u}{v}\right) \pm \tfrac{1}{2}\sqrt{\tfrac{u^2}{v^2} + a};$$

in the present instance, $a = 800$, $u = 100$, $v = 5$, which gives

$$p = \tfrac{1}{2}(800 + 10) \pm \tfrac{1}{2}\sqrt{100 + 800} = 405 \pm 15;$$

the lower sign being employed, we have $p = 390$.

In fact, on substituting this number in the formula (1) we find that he neither wins nor loses money, although the number of unfavourable events has been greater by twenty than the number of favourable ones. The other root of the equation points out another limit, beyond which if the number of favourable events increase, he will lose money.

In the problem just examined, each stake depends on that which immediately precedes it. Others may be proposed, in which it is made to depend on the result of two or more of the events which immediately precede, such is the following one:

A gamester begins by staking a given sum u, and regulates his succeeding bets in this manner; if one of the two immediately antecedent to that which he is at any time making, have been gained, and the other lost, the new stake is the same as that last made; but if both those bets have been gained, he diminishes his stake by the sum v; and if both have been lost, he increases it by the same sum v. The first stake being u, the first gain may be represented by $u(-1)^a$; and as there has only been one bet yet made, the second cannot be affected by the law prescribed; it will therefore be the same as the first, and the profit arising from it will be $u(-1)^b$. The third stake depending on the two former, we must add to the quantity u some function of a and b multiplied by v, which shall vanish if a and b are one odd and the other even, and which shall become -1 when they are both even, and $+1$ when they are both odd. Such is the function $-\frac{(-1)^a+(-1)^b}{2}$; the profit from the third stake may therefore be represented by

$$\left\{u - v\left(\frac{(-1)^a+(-1)^b}{2}\right)\right\}(-1)^c;$$

and in a similar manner, we may find that which arises from the fourth is expressed by

$$\left\{ u - v \left(\frac{(-1)^a + (-1)^b}{2} + \frac{(-1)^b + (-1)^c}{2} \right) \right\} (-1)^d.$$

The mode of continuation is sufficiently obvious, and by arranging all these profits together, we have

$$u \left\{ (-1)^a \right\}$$

$$u \left\{ (-1)^b \right\}$$

$$\left\{ u - v \left(\frac{(-1)^a + (-1)^b}{2} \right) \right\} (-1)^c$$

$$\left\{ u - v \left(\frac{(-1)^a + (-1)^b}{2} + \frac{(-1)^b + (-1)^c}{2} \right) \right\} (-1)^d$$

$$\left\{ u - v \left(\frac{(-1)^a + (-1)^b}{2} + \frac{(-1)^b + (-1)^c}{2} + \frac{(-1)^c + (-1)^d}{2} \right) \right\} (-1)^e$$

$$\left\{ \begin{array}{l} u + v \left(\frac{(-1)^a + (-1)^b}{2} + \frac{(-1)^b + (-1)^c}{2} + \frac{(-1)^c + (-1)^d}{2} \right. \\ \qquad\qquad \left. + \frac{(-1)^d + (-1)^e}{2} \right) \end{array} \right\} (-1)^f.$$

&c. &c.

The part which depends on u is easily determined to be $(p - q)\,u$, where p and q are the number of successful and of unsuccessful cases, the remaining part consists of $-\frac{v}{2}$ multiplied by a function of the quantities a, b, c, &c. In order to abridge as much as possible, I shall write a instead of $(-1)^a$, b instead of $(-1)^b$, and so on; so that $a\,b$ will represent

$(-1)^a \times (-1)^b$, or $(-1)^{a+b}$; observing this, the factor multiplying $-\frac{v}{2}$ will be expressed thus:

$$\begin{array}{l}
ac + bc \\
ad + 2bd + cd \\
ae + 2be + 2ce + de \\
af + 2bf + 2cf + 2df + ef \\
ag + 2bg + 2cg + 2dg + 2eg + fg, \\
\qquad \&c. \qquad \&c.
\end{array}$$

If to these were added the two series $ab + bc + cd +$ &c. and $a(b + c + d + e +$ &c.$)$ the sum would be equal to twice the sum of all the products, taken two by two of the quantities a, b, c, &c. (and since these quantities represent $(-1)^a$, $(-1)^b$, &c. we have found in the last problem that it is equal to $\left(\frac{(p-q)^2-(p+q)}{2}\right)$. The part depending on v is therefore

$$-\frac{v}{2}\left\{(p-q)^2-(p+q)-(-1)^{a+b}-(-1)^{b+c}-(-1)^{c+d}-\&c. \right.$$
$$\left. -(-1)^a\left((-1)^b+(-1)^c+(-1)^d+\&c.\right)\right\}$$

The second of these series is evidently equal to

$$(-1)^a\left(p-q-(-1)^a\right)=(-1)^a(p-q)-1\,;$$

the other series

$$(-1)^{a+b}+(-1)^{b+c}+(-1)^{c+d}+(-1)^{d+e}+\&c.$$

cannot be determined merely by knowing how many of the numbers a, b, c, &c. are odd, and how many are even.

Still, however, the knowledge of the order in which the events take place is not absolutely required. It will be sufficient to know how many changes from odd to even, and *vice versa*, occur between the quantities a, b, c, d, e, &c.; for if there is no change in going from a to b, the first term will be $+1$; but if there is a change it will be -1; and generally -1 will occur as often as there is a change from odd to even, or from even to odd in the series

$$a,\ b,\ c,\ d,\ e,\ \&c.$$

If, therefore, k denote the number of transitions, since the number of terms is $p+q-1$, the value of the series will be

$$p+q-2k-1;$$

and the whole value of the part dependent on v will be

$$-\frac{v}{2}\left\{(p-q)^2-(p+q)-(p+q)+2k+1-(-1)^a(p-q)+1\right\}$$

and the profit on the whole number of bets is

$$W=(p-q)u-\frac{v}{2}\left\{(p-q)^2-2(p+q)+2k+2-\right.$$
$$\left.(-1)^a(p-q)\right\} \qquad (2)$$

if $p=q$, then

$$W_1=+v(2p-k-1) \qquad (3)$$

k in the first of these expressions may vary from 0 to $p+q-1$, and in the second from 1 to $2p-1$.

In order to compare this with an example, let the following series of bets be made *plus* or *minus*, denoting their success or failure :

$+ 100$
$+ 100$
$+ 90$
$- 80$
$+ 80$
$+ 80$
$+ 70$

The profit is 440. Here $p = 6$, $q = 1$, $k = 2$, $v = 10$, $u = 100$; and these values being substituted in (2) give

$$W = 440.$$

The next case I shall examine is one in which the amount of each stake depends on the result of the three preceding events.

Let the law by which the stakes are regulated be such, that when the three preceding have been won, a certain sum v is taken from the last stake; if they have been lost, the sum v is added to the last; but if only two of these three have been gained, or only two lost, in the first case one-third of v is subtracted, and in the second case is added to the last stake.

The first stake being u, the three first bets will be represented by $u(-1)^a$, $u(-1)^b$, $u(-1)^c$. In order to fulfil the conditions of the question, the fourth must be increased by the quantity $-v\,\frac{(-1)^a + (-1)^b + (-1)^c}{3}$; and the profit resulting from it is represented by

$$\left\{ u - v\,\frac{(-1)^a + (-1)^b + (-1)^c}{3} \right\} (-1)^d$$

that of the fifth is equally expressed by

$$\left\{ u - u \left(\frac{(-1)^a + (-1)^b + (-1)^c}{3} + \frac{(-1)^b + (-1)^c + (-1)^d}{3} \right) \right\} (-1)^e$$

The succeeding terms need not be written down, as the law is obvious. In this, as in the former questions, the co-efficient of u is $p - q$; and if we employ the same abridgment of a for $(-1)^a$, &c. as in the last, the co-efficient of $-\frac{v}{3}$ is

$$\begin{array}{l} ad + bd + cd \\ ae + be + 2ce + de \\ af + 2bf + 3cf + 2df + ef \\ ag + 2bg + 3cg + 3dg + 2eg + fg \\ \qquad\quad \&c. \quad \&c. \end{array}$$

This series, which only differs from three times the sum of all the products, two by two of the quantities a, b, c, &c. by the omission of certain parts, may be called Q. On examining what parts are omitted, it will be found that

$$\begin{aligned} &\mathrm{Q} + 2a\big(\mathrm{S}(a+b+..) - a - b - c\big) + bd \\ &\quad + b\big(\mathrm{S}(a+b+..) - a - b - c\big) \\ &\quad + (ce + df + cg + ..) + 2(cd + de + ef + ..) \\ &\quad + 3(ab + ac + bc) = 3\,\mathrm{S}(ab + ..) \\ &\quad = 3\,\frac{(p-q)^2 - (p+q)}{2}. \end{aligned}$$

This equation becomes, by the mutual destruction of certain terms

$$\begin{aligned} &\mathrm{Q} + 2a\,\mathrm{S}(a+b+..) + b\,3(a+b+..) + (ac + bd + cc + ..) \\ &\quad + 2(bc + cd + ..) - 3 = 3\,\frac{(p-q)^2 - (p+q)}{2}. \end{aligned}$$

Hence

$$Q = 3\,\frac{(p-q)^2-(p+q)}{2} - 2\,(-1)^a\,(p-q) - (-1)^b\,(p-q) + 3 -$$

$$-\,(a\,c + b\,d + c\,d + \,.\,.) - 2\,(b\,c + c\,d + \,.\,.\,.).$$

The latter of these series is

$$\left\{ (-1)^{a+b} + (-1)^{b+c} + (-1)^{c+d} + \ldots \right\} - (-1^{a+b} =$$

$$= p + q - 2\,k - 1 - (-1)^{a+b},$$

as we found in the last question, k being the number of changes from even to odd, or *vice versa*, in the series $a, b, c, \,.\,.$ The value of the series

$$(-1)^{a+c} + (-1)^{b+d} + (-1)^{c+e} + \,.\,.$$

is not determined without some other data. If, however, we are acquainted with the number of alterations from odd to even, and the contrary, in the series

$$a\,c \quad b\,d \quad c\,e \quad d\,f \quad .\;.\;.$$

we can assign its value; let the number be l, then the series in question is equal to $p + q - 2\,l - 2$: These substitutions and the necessary reductions being made, we have

$$Q = (p-q)\left\{3\,\frac{p-q}{2} - 2\,(-1)^a - (-1)^b\right\} - (p+q)\,\frac{q}{2}$$

$$+ 7 + 2\,l + 4\,k + 2\,(-1^{a+b}$$

and W the profit on $p + q$ events is

$$W = (p-q)\left\{u - \frac{v}{3}\left(3\,\frac{p-q}{2} - 2\,(-1)^a - (-1)^b\right)\right\}$$

$$-\,\frac{v}{3}\left\{7 - (p+q)\,\frac{q}{2} + 2\,l + 4\,k + 2\,(-1)^{a+b}\right\} \qquad (4)$$

if $p = q$

$$W_1 = -\,\frac{v}{3}\left\{7 - 9\,p + 2\,l + 4\,k + 2\,(-1)^{a+b}\right\} \qquad (5)$$

In the first of these expressions, k may vary from 0 to $p+q-1$, and l from 0 to $p+q-2$; in the second k may vary from 1 to $2p-1$, and l may vary from 0 to $2p-2$.

The last question which I shall examine is one in which each stake depends on all those which precede it.

A gamester stakes a certain sum u on an event whose chance of happening is one-half. He regulates each succeeding stake in this manner: To the constant sum u he adds the nth part of all his previous winnings; or if he has lost by the previous stakes, he subtracts the nth part of his loss, it is proposed to find his profit * at the termination of $p+q$ bets.

The first stake being u, the profit is $u(-1)^a$; the nth part of this added to u, or $u+\frac{u}{n}(-1)^a$ will constitute the second, and the profit will be $u(-1)^b+\frac{u}{n}(-1)^a(-1)^b$, the nth part of these two, or $\frac{u}{n}\left((-1)^a+(-1)^b\right)+\frac{u}{n^2}(-1)^a(-1)^b$ added to the quantity u, will be the third stake, and the profit on the determination of the third event will be

$$u(-1)^c+\frac{u}{n}\left((-1)^a+(-1)^b\right)(-1)^c+\frac{u}{n^2}(-1)^a(-1)^b(-1)^c;$$

the

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* The language of analysis is so much more general than that in which we usually convey our thoughts, that it is almost impossible to make the latter keep pace with the former. This is more particularly manifest when we are treating of games of chance. The words profit, winning, gain, &c. must, if we wish to avoid perpetual repetition, frequently be understood to comprehend their very opposites.

the nth part of the sum of these three profits added to u, and multiplied by $(-1)^d$, gives for the profit on the fourth event

$$u(-1)^d + \frac{u}{n}\left((-1)^a(-1)^d + (-1)^b(-1)^d + (-1)^c(-1)^d\right)$$

$$+ \frac{u}{n^2}\left((-1)^a(-1)^b(-1)^d + (-1)^a(-1)^c(-1)^d + (-1)^b(-1)^c(-1)^d\right)$$

$$+ \frac{u}{n^3}(-1)^a(-1)^b(-1)^c(-1)^d.$$

From these expressions, continued a few steps further if necessary, it appears that the sum of all the profits W is equal to

u multiplied by the sum of all the quantities

$(-1)_a$, $(-1)^b$, $(-1)^c$, . .

$+ \frac{u}{n}$ multiplied by the sum of all the products, two by two of those quantities.

$+ \frac{u}{n^2}$ multiplied by the sum of all the products, three by three of the same quantities.

$+ \frac{u}{n^3}$ multiplied by the sum of all the products, four by four of the same quantities.

&c. &c.

Hence $\frac{1}{n}$ W $=$ the sum of all the co-efficients, except the first of the terms of an equation, whose roots are $\frac{(-1)^a}{n}$, $\frac{(-1)^b}{n}$, $\frac{(-1)^c}{n}$, &c. multiplied by u; but by the conditions of the problem, p of the quantities a, b, c, are even, and q of them odd numbers, or p of the roots of the equation are of the form $\frac{-1}{n}$, and q of the form $\frac{+1}{n}$, and the equation itself is the developement of $\left(x+\frac{1}{n}\right)^p \left(x-\frac{1}{n}\right)^q = 0$,

the sum of all its co-efficients except the first may be found by making $x = 1$, and subtracting unity, this gives

$$\frac{1}{n} \mathrm{W} = \left(1+\frac{1}{n}\right)^p \left(1-\frac{1}{n}\right)^q u - u,$$

$$\text{or} \quad \mathrm{W} = n\left(\frac{n+1}{n}\right)^p \left(\frac{n-1}{n}\right)^q u - n\,u\,; \qquad (6)$$

if $p = q$ this becomes

$$\mathrm{W} = \frac{(n^2-1)^p}{n^{2p-1}}\, u - n\,u \qquad (7)$$

if $n = 1$, whatever be the numbers p and q, the loss will be equal to u (unless at the same time $q = 0$), for in that case as soon as the first unfavourable event happens, the player loses not only all he had previously won, but also the sum u besides; and since, by the conditions of the play, he must subtract from u the nth part of all his former loss, which since $n = 1$ is $-u$, his next stake must be $u - u$, or zero; so that in fact if $n = 1$, the first unfavourable decision terminates the game.

I shall now proceed to show how a similar mode of reasoning may be applied to cases where the number of events which happen at each step are more than two.

Suppose a person draw a number of balls in succession from an urn, containing balls numbered 1, 2, 3, 4, . . k, there being many of each kind: he begins by staking the sum u, and if he draw a ball marked one, he receives n_1 times his stake, if the ball be marked two, he receives n_2 times his stake, and so on. The next stake is thus regulated, supposing the ball last drawn to have been marked i, he adds to the last stake the sum $v n_i$: any of the numbers n_1 n_2, . . may be negative, if he has drawn p balls marked n_1, q marked n_2, r marked n_3, and so on, what is the amount of his winning?

Let $\alpha, \beta, \gamma, \ldots$ be the kth roots of unity, the expression

$$S_a = \frac{\alpha^a + \beta^b + \gamma^a + \cdot\cdot}{k}$$

is always equal to zero, except when a is a multiple of k; also let

$$P_a = n_1\, S_a + n_2\, S_{a+1} + n_3\, S_{a+2} + \cdot\cdot + n_k\, S_{a+k-1}\,;$$

then P_a will in every case reduce itself to one of the quantities $n_1, n_2, n_3, \ldots n_k$.

With the aid of these considerations, we can express the amount of the stake at any particular step; his first is u, and whatever be the kind of ball drawn, his profit is always expressed by $u\,P_a$, according to the form of a. In consequence of this first determination, he adds to u the quantity $v\,P_a$, which sum $u + v\,P_a$ forms his second stake; the number of the second ball determines the amount of his second profit, which may be expressed thus:

$$(u + v\,P_a)\,P_b,$$

without at all determining the form of a, the third stake will be $u + v\,P_a + v\,P_b$ and the profit arising from it is

$$(u + v\,P_a + v\,P_b)\,P_c\,;$$

that in the fourth event is

$$(u + v\,P_a + v\,P_b + v\,P_c)\,P_d.$$

the sum of all the profits will be

$$\begin{aligned}
&u\,P_a \\
&u\,P_b + v\,(P_a)\,P_b \\
&u\,P_c + v\,(P_a + P_b)\,P_c \\
&u\,P_d + v\,(P_a + P_b + P_c)\,P_d \\
&u\,P_e + v\,(P_a + P_b + P_c + P_d)\,P_e .
\end{aligned}$$

It is easily perceived that the co-efficient of u is equal to that of the second term, and the co-efficient of v is equal to that of the third term of the equation

$$(x + P_a)\,(x + P_b)\,(x + P_c)\,(x + P_d)\ldots = 0;$$

but of the quantities* P_a, P_b, P_c, . . p are equal to n_1, q are equal to n_2, r are equal to n_3, &c. this equation is therefore

$$(x + n\ )^p\,(x + n_2)^q\,(x + n_3)^r\,\ldots = 0;$$

and the co-efficient of the second term, in its developement, is

$$p\,n_1 + q\,n_2 + r\,n_3 + \ldots$$

whilst that of the third term is

$$\frac{p\cdot p-1}{2}\,n_1{}^2 + \frac{q\cdot q-1}{2}\,n_2{}^2 + \frac{r\,.\,r-1}{2}\,n_3{}^2 + \,.\,.$$

$$+\frac{pq}{1}\,n_1\,n_2 + \;+\frac{pr}{1}\,n\;\;n_3 + \frac{qr}{1}\,n_2\,n_3 + \ldots$$

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* This notation has been employed by Mr Herschel, in a paper in the Philosophical Transactions for the year 1818 on circulating functions.

so that the whole profit is

$$W = u\,(p\,n_2 + q\,n \;+ r\,n_3 + \ldots) + v\left\{\frac{p\cdot p-1}{2}\,n_1{}^2 + \frac{q\cdot q-1}{2}\,n_2{}^2 + \ldots\right.$$
$$\left. + \frac{p\,q}{1}\,n_1\,n_2 + \frac{p\,r}{1}\,n_1\,n_3 + \ldots\right\} \qquad (8)$$

in the case of $n_1 = 1$, $n_2 = -1$, $n_3 = 0$. This formula reduces itself to (1).

Supposing the urn in the last question filled with the same balls, and a person drawing out one receives n_1, n_2, n_3, . . times the sum u, according to the number of the ball drawn; and on the second drawing he receives n_1, n_2, n_3, . . times the sum of $u +$ the profit by the last drawing: and generally on extracting any ball he receives n_1, n_2, n_3,. . times the sum of $u +$ the amount of the profit on all the preceding events, if the number of times each of the balls marked 1, 2, 3, . . are drawn, be respectively denoted by p, q, r, . . what is the whole profit?

Adopting the same notation as in the last problem, $P_a = n_1\,S_a + n_2\,S_{a+1} + n_3\,S_{a+2} + \ldots$ will represent either n_1, n_2, n_3, . . and the first profit is $u\,P_a$, the second is $(u + u\,P_a)\,P_b$; that in the third is $(u + u\,P_a + u\,P_b + u\,P_a\,P_b)\,P_c$, and the sum of all the profits is

$$\begin{array}{l} u\,P_a \\ u\,P_b + u\,P_a\,P_b \\ u\,P_c + u\,P_a\,P_c + u\,P_b\,P_c + u\,P_a\,P_b\,P_c, \\ \qquad\&\text{c.} \qquad \&\text{c.} \qquad \&\text{c.} \end{array}$$

On comparing this with the fourth problem which was solved, it appears that

$$\mathrm{W} = u \times \text{the sum of all the co-efficients except the first of the equation } (x + \mathrm{P}_a)(x + \mathrm{P}_b)(x + \mathrm{P}_c)\ldots = 0,$$

or $\mathrm{W} = u(1 + \mathrm{P}_a)(1 + \mathrm{P}_b)(1 + \mathrm{P}_c)\ldots - u$;

but p of the quantities $\mathrm{P}_a, \mathrm{P}_b, \mathrm{P}_c, \ldots$ are equal to n_1, q of them to n_2, r of them to $n_3, \ldots$ This equation, therefore, becomes

$$\mathrm{W} = u(1 + n_1)^p (1 + n_2)^q (1 + n_3)^r \ldots - u \qquad (9)$$

if $n_1 = \frac{1}{n}$, $n_1 = -\frac{1}{n}$, $n_3 = 0$, and $u = n u$. This coincides with (6).

As an example, suppose an urn filled with balls of three colours, white, black, and red, and that the person who draws them out may name any sum he chose prior to each extraction; if he draw a white ball, the sum he named is paid to him; if a black, he loses one-half of it; and if a red one, he loses one-third of that sum. And suppose he regulates the sum named in the following manner, beginning with naming u whenever he has drawn a white ball, he adds the whole of his previous winnings to the sum u; but if he has drawn a black one, he adds only half his profits to the sum u; and if the ball last extracted from the urn was red, he adds one-third of all his profits to the same sum u. He has drawn out p white, q black, and r red balls, what is the amount of his profit or loss?

In this case $n_1 = 1$, $n_2 = -\frac{1}{2}$, $n_3 = -\frac{1}{3}$, $n_4 = n_5 = .. = 0$

and we have

$$W = u\,2^p \cdot \frac{1}{2^q} \cdot \frac{2_r}{3^r} - u = \frac{2^{p-q+r}}{3^r}\,u - u\,.$$

The questions examined in the preceding pages afford an instance of the immediate application of some very abstract propositions of analysis to a subject of constant occurrence, which being as far as I have been able to discover, hitherto untouched, and also requiring reasoning of rather an unusual nature, I have preferred treating in particular instances, instead of investigating it in its most general form. The principles on which similar problems should be attempted, are first by means of some combinations of the roots of unity to represent the stake after the decision of any number of events, and then by means of any known theorems respecting the roots of equations, to ascertain the sums of the series which present themselves in the result.